Model Predictive Control

Theodor Chakhachiro

Fall 2020

Content

Following Prof. Stephen Boyd [1]

1. Linear Convex Optimal Control
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- The general formulation of the problem is as follows:

min

$$J = \sum_{t=0}^{\infty} \ell(x(t), u(t))$$

subject to

$$u(t) \in \mathcal{U}$$
$$x(t) \in \mathcal{X}$$
$$x(t+1) = Ax(t) + Bu(t)$$
$$x(0) = z$$

- Two options for the optimization variable, either consider both $u(t)$ and $x(t)$ as the optimization variables or just $u(t)$

- Choose $u(t)$ to minimize current stage cost over allowed control input action:

$$u(t) = argmin_w \ \ \{\ell(x(t), w) | w \in \mathcal{U}, Ax(t) + Bw \in \mathcal{X}\}$$

- This type of control in not optimal since if $||A||_2$ is small, $x(t+1)$ does not depend much on $x(t)$
- This model is thus subject to a "fading memory" problem

$$V(z) = inf\{\ell(z,w) + V(Az+Bw) | w \in \mathcal{U}, Az+Bw \in \mathcal{X}\}$$

- $V(z)$ is called the value function or the Bellman function
- $V(z)$ is a convex function
- The optimal control input is given by:

$$u^*(t) = \phi(x(t)) = argmin\{\ell(x(t),w) + V(Ax+Bw)\}$$

- The cost function is given by:

$$\ell(x(t), u(t)) = x(t)^T Q x(t) + u(t)^T R u(t)$$

- Where $Q \succeq 0, R \succ 0, \mathcal{U} \subseteq \mathbb{R}^m, \mathcal{X} \subseteq \mathbb{R}^n$
- The value function is obtained as:

$$V(z) = z^T P z$$

- Where $P \in \mathbb{S}^n_{++}$, is obtained by solving the Algebraic Ricatti Equation:

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

- The optimal control input is found to be:

$$u^*(t) = K x(t)$$

- Where the feedback gain K is:

$$K = -(R + B^T P B)^{-1} B^T P A$$

Linear Convex Optimal Control

Linear Quadratic Regulator (LQR) cont.

- Looking back at the Algebraic Ricatti Equation, we can notice that when solving for P, we have a Schur complement in the equation

Schur complement in ARE

- Algebraic Ricatti Equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

- Solving for P:

$$P = Q + S$$

Where S is the Schur complement of the following matrix:

$$\begin{bmatrix} R + B^T PB & B^T PA \\ A^T PB & A^T PA \end{bmatrix}$$

$$S = A^T PA - A^T PB(R + B^T PB)^{-1}B^T PA$$

- Now instead of having an infinite dimensional convex problem, we specify a horizon T:

$$\min \quad J = \sum_{t=0}^{T-1} \ell(x(t), u(t))$$

$$\text{subject to} \quad \begin{aligned} u(t) \in \mathcal{U},\ x(t) \in \mathcal{X},\ t = 0, ..., T \\ x(t+1) = Ax(t) + Bu(t),\ t = 0, ..., T-1 \\ x(0) = z,\ x(T) = 0 \end{aligned}$$

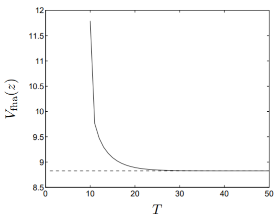
- For $t \geq T$, $u(t) = 0$
- Thus, we are working in a finite dimensional subspace over which we will obtain a sub-optimal control input $u^*(t)$

Finite Horizon Approximation

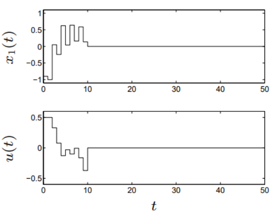
LQR with In/finite Horizon

Example

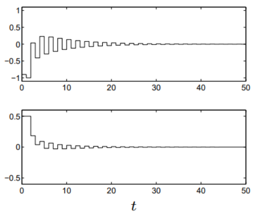
- $x(t) \in \mathbb{R}^3$, $u(t) \in \mathbb{R}^2$ and A, B are randomly chosen
- $\mathcal{X} = \{v|\ ||v||_\infty \leq 1\}$, $\mathcal{U} = \{w|\ ||w||_\infty \leq 0.5\}$
- Stage Cost is defined as: $\ell(x(t), u(t)) = ||v||^2 + ||w||^2$
- $x(0) = [0.9, -0.9, 0.9]^T$



(a) Cost versus horizon



(b) $x_1(t)$, $u_1(t)$ for $T = 10$



(c) $x_1(t)$, $u_1(t)$ for $T = \infty$

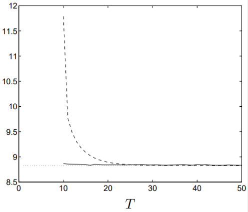
- Also known as Receding Horizon Control (RHC), Finite Look-ahead Control, Dynamic Linear Programming
- Early implementations included control of chemical processes, supply chain management and revenue management
- These early implementations of MPC only revolved around slow dynamic processes

- For each time t, the following planning problem is solved:

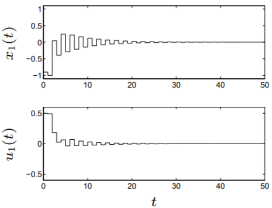
min
$$J = \sum_{\tau=t}^{t+T} \ell(x(\tau), u(\tau))$$

subject to
$$u(\tau) \in \mathcal{U},\ x(t) \in \mathcal{X},\ \tau = t, ..., T + t$$
$$x(\tau + 1) = Ax(\tau) + Bu(\tau),\ \tau = t, ..., t + T - 1$$
$$x(t + T) = 0$$

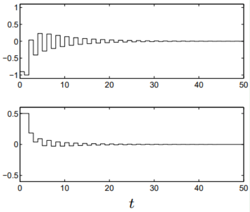
Example



(d) Comparison between MPC and Finite Horizon Approximation



(e) $x_1(t),\ u_1(t)$ *for* $T = 10$



(f) $x_1(t),\ u_1(t)$ *for* $T = \infty$

- Relaxing the final state condition from $x(t+T)=0$ to $x(t+T)=\hat{V}(x(t+T))$
- Using current plan for $K>1$ steps ahead instead of 1
- Convert hard constraints to violation penalties to avoid problem of planning problem infeasibility

- MPC problem is highly structured: Hessian is block diagonal and equality constraint matrix is block banded
- Can be solves in order of $T(n+m)^3$ flops using an interior point method

Fast MPC Implementations

- Fast MPC implementations can solve high dimensional problems in matter of milliseconds
- This is done by using a warm-start. This means that the initial guess for the problem is chosen by solving for the optimal value of a related/simplified optimization problem
- Other tricks include limiting the number of Newton steps

Example

problem size			QP size		run time (ms)	
n	m	T	vars	constr	fast mpc	SDPT3
4	2	10	50	160	0.3	150
10	3	30	360	1080	4.0	1400
16	4	30	570	1680	7.7	2600
30	8	30	1110	3180	23.4	3400

Figure: Fast MPC results in C

Supply Chain Management

Problem Structure

- n nodes for storing goods (tanks in water treatment facility)
- m links between nodes (pipes)
- $x_i(t)$ is the amount of commodity at node i, in period t (amount of water in each tank)
- $u_j(t)$ is the amount of commodity at node j, in period t (amount of water in each pipe)
- Incoming and outgoing node incidence matrices (flow direction in each pipe):

$$A_{ij}^{in(out)} = \begin{cases} 1 & \textit{if link } j \textit{ enters (exits) node } i \\ 0 & \textit{otherwise} \end{cases}$$

- Dynamics are linear (ignoring spoilage):

$$x(t+1) = x(t) + A^{in}u(t) - A^{out}u(t)$$

- The general formulation of the problem is as follows:

$$\min \quad \sum_{t=0}^{\infty}(S(u(t)) + W(x(t)))$$

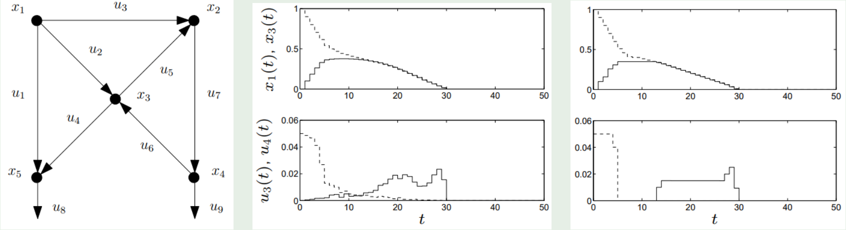
$$\text{subject to} \quad \begin{aligned} 0 \leq x_i(t) &\leq x_{max} \\ 0 \leq u_j(t) &\leq u_{max} \\ A^{out}u(t) &\preceq x(t) \end{aligned}$$

- $(u(t))$ is the shipping cost and $W(x(t))$ is the storage cost. A fixed cost can also be added which might nonlinearize the problem

Supply Chain Management

Example

- $n = 5,\ m = 9$
- $x_{max} = 1,\ u_{max} = 0.05$
- $W(x(t)) = \sum_{i=1}^{n}(x_i(t) + x_i(t)^2)$
- $W(x(t)) = \sum_{j=1}^{7} u_j(t) - \sum_{j=8}^{9} u_j(t)$
- $x(0) = [1, 0, 0, 1, 1]^T,\ V(z) = 68.2$ and $V_{MPC} = 69.5$



(a) Node graph *(b) MPC results with $T = 5$* *(c) Optimal results*

MPC Variations

- Time varying costs, dynamics, constraints
- Coupled state and input constraints
- Slew rate constraints: $e.g., ||u(t+1) - u(t)||_\infty \leq \Delta u_{max}$
- Stochastic Control

Brief Introduction to Stochastic MPC

- Linear dynamical system:

$$x(t+1) = Ax(t) + Bu(t) + \omega(t)$$

- x_0, $\omega_0, ..., \omega_{T-1}$ are random variables
- Objective function:

$$J = E(\sum_{t=0}^{T-1}(\ell_t(x(t), u(t)) + \ell_T(x(T))))$$

- J depends on control policies $\phi_0, ..., \phi_{T-1}$, which are the problem variables
- Choose the control policy $\phi(t)$ of choosing $u(t)$

References I

[1] Stephen Boyd, *Lecture notes in convex optimization*, https://web.stanford.edu/class/ee364b/lectures/mpc_slides.pdf, July 2008.